جامعة الجزيرة الخاصة
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Signal processing
معالجة الاشارة

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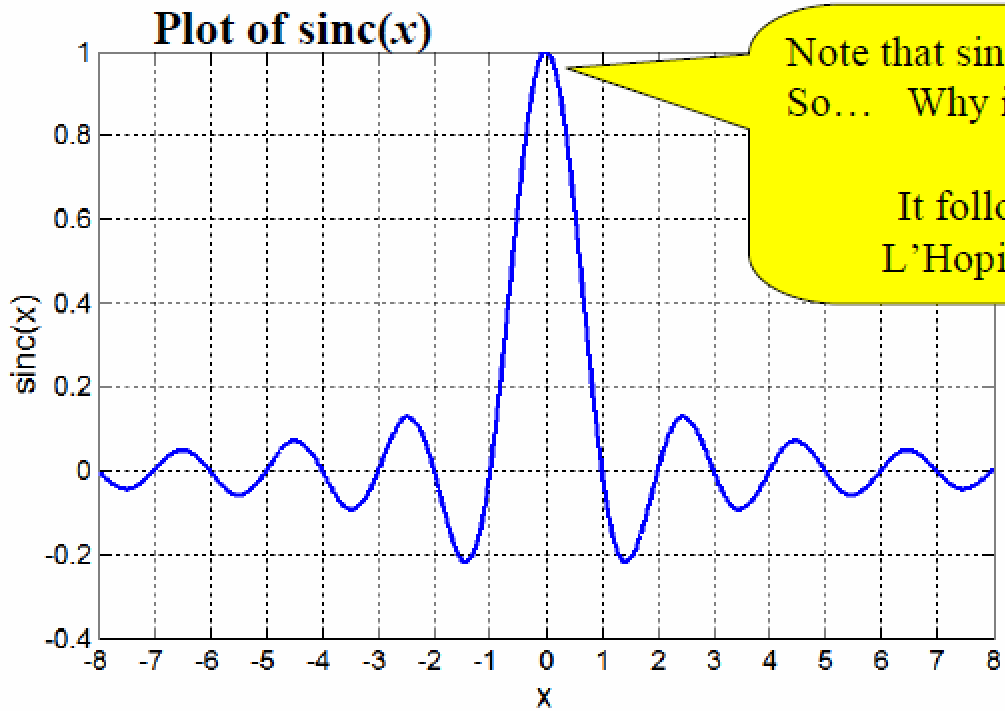
Definition of "Sinc" Function

The result we just found had this mathematical form: $P_\tau(\omega) = \dfrac{2\sin\left(\dfrac{\omega\tau}{2}\right)}{\omega}$

This kind of structure shows up frequently enough that we define a special function to capture it:

Define: $\text{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$



Note that sinc(0) = 0/0.
So… Why is sinc(0) = 1?

It follows from L'Hopital's Rule

With a little manipulation we can re-write the FT result for a pulse in terms of the sinc function:

Recall:

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Need π times something…

$$P_\tau(\omega) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega} = \frac{2\sin\left(\frac{\pi}{\pi}\frac{\omega\tau}{2}\right)}{\omega} = \frac{2\sin\left(\pi\frac{\omega\tau}{2\pi}\right)}{\omega}$$

Now we need the same thing down here as inside the sine…

$$= \pi\frac{\tau}{2\pi}\frac{2\sin\left(\pi\frac{\omega\tau}{2\pi}\right)}{\pi\frac{\tau}{2\pi}\omega} = \tau\frac{\sin\left(\pi\frac{\omega\tau}{2\pi}\right)}{\pi\frac{\omega\tau}{2\pi}} = \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

$$P_\tau(\omega) = \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

Table of Common Fourier Transform Results

We have just found the FT for two common signals…

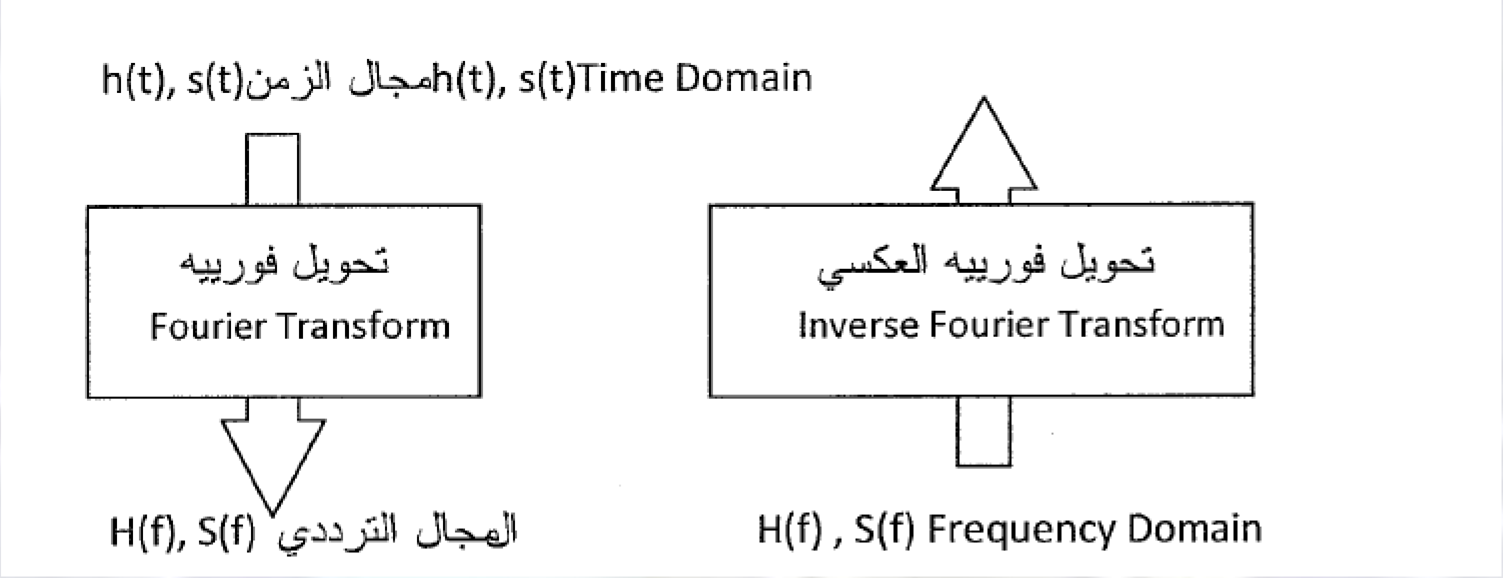
$$x(t) = e^{-bt}u(t) \longleftrightarrow X(\omega) = \frac{1}{b + j\omega}$$

$$p_\tau(t) = \begin{cases} 1, & -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0, & otherwise \end{cases} \longleftrightarrow P_\tau(\omega) = \tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

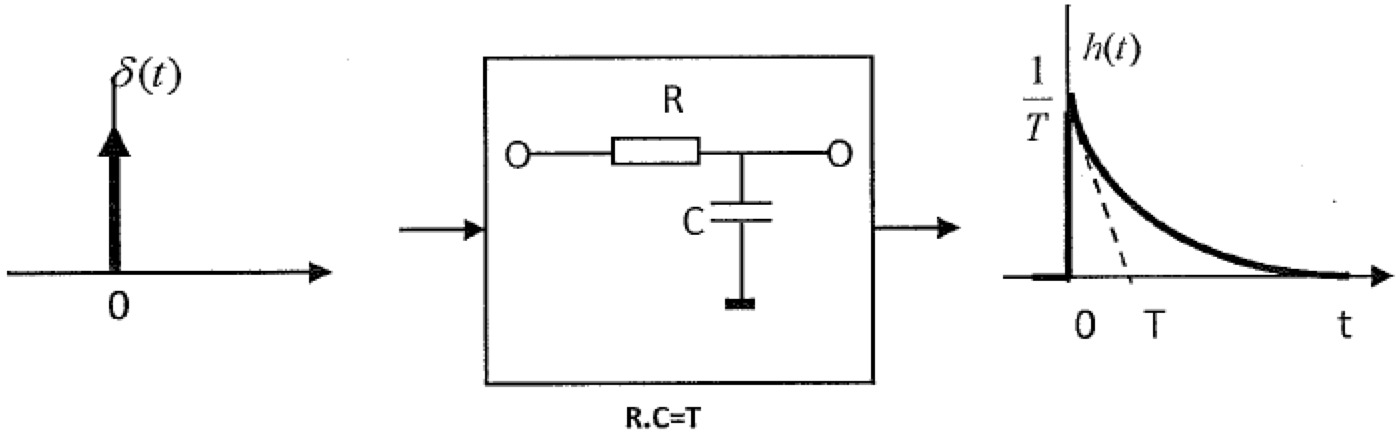
تحويل فورييه Fourier Transform وتحويل فورييه العكسي Inverse Fourier Transform:

تحويل فورييه:

- تابع النقل: للحصول على تابع نقل جملة LTI انطلاقاً من تابع استجابتها النبضية $h(t)$ تستخدم علاقة تحويل فورييه الآتية: $$H(f)=\int_{-\infty}^{+\infty} h(t).e^{-j2\pi ft}.dt$$
- طيف الإشارة: للحصول على طيف الإشارة يطبق تحويل فورييه على تابع الإشارة في مجال الزمن $s(t)$، أي: $$S(f)=\int_{-\infty}^{+\infty} s(t)\,e^{-j2\pi ft}.dt$$



مثال تطبيقي:لنأخذ تحويل فورييه للنبضة الآسية. إن دارة RC كجملة LTI تستجيب على نبضة ديراك بما يدعى بتابع الاستجابة النبضي $h(t)$ وفق العلاقة: $h(t)=\frac{1}{T}.u(t).e^{-\frac{t}{T}}$ وذلك كما في الشكل الآتي.



دارة RC كجملة LTI وتابع استجابتها النبضي

المطلوب استنتاج تابع النقل وتحليل علاقته:

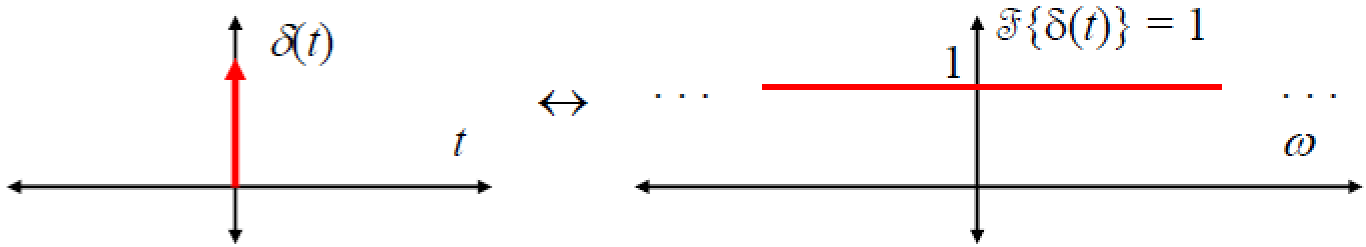
الحل: إيجاد تابع النقل $H(f)$:

$$H(f)=\frac{1}{T}.\int_{0}^{+\infty}e^{-\frac{t}{T}}.e^{-j2\pi ft}.dt=\frac{1}{T}.\int_{0}^{+\infty}e^{-(\frac{1}{T}+j2\pi f)t}.dt$$

$$=\frac{1}{T}.\frac{-1}{\frac{1}{T}.+j2\pi f}[e^{-(\frac{1}{T}+j2\pi f)t}]_{0}^{+\infty}=\frac{-1}{1+j2\pi fT}[e^{-(\frac{1}{T}+j2\pi f)t}]_{0}^{+\infty}$$

$$H(f)=\frac{1}{1+j2\pi fT}$$

$$\mathfrak{F}\{\delta(t)\} = \underbrace{\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega \cdot 0}}_{\text{Sifting property}} = 1 \quad \Rightarrow \quad \boxed{\delta(t) \leftrightarrow 1}$$

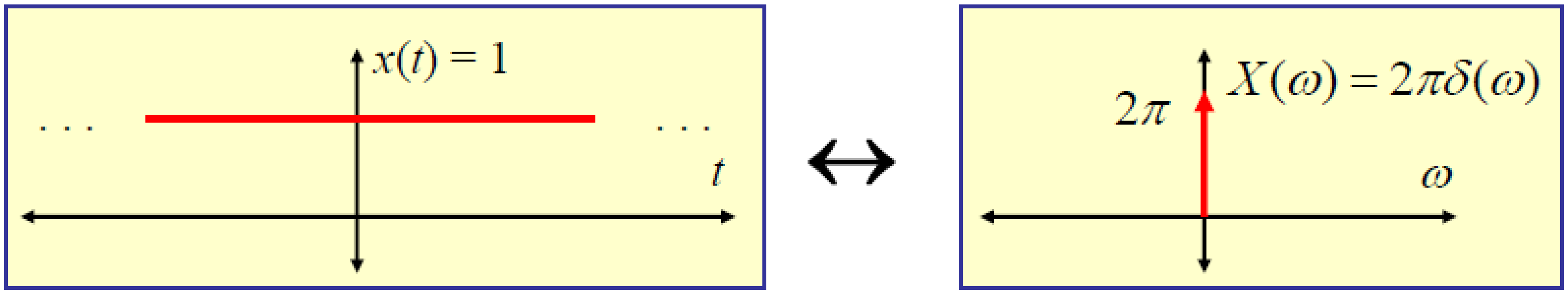


Now we can use the duality property to get another FT Pair:

$$\delta(t) \leftrightarrow 1$$

$$1 \leftrightarrow 2\pi\delta(\omega)$$

So we now know:



"A DC signal" has FT concentrated at 0 Hz

DC = 0 Hz

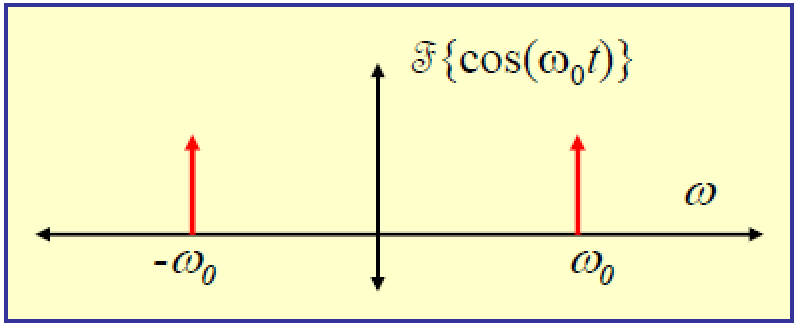
Now we can get ***another*** pair by using this last result and the real modulation property:

$$1 \leftrightarrow 2\pi\delta(\omega)$$

Multiply by cosine

Frequency Shift Up & Down

$$\cos(\omega_0 t)\times 1 \leftrightarrow \frac{1}{2}\left[2\pi\delta(\omega+\omega_0)+2\pi\delta(\omega-\omega_0)\right]$$



Note: This says you only need the components at $+\omega_0$ and $-\omega_0$ (i.e., $\exp\{j\omega_0 t\}$ and $\exp\{-j\omega_0 t\}$) to build $\cos(\omega_0 t)$

Can do similar thing for sine:

$$\cos(\omega_0 t) \leftrightarrow \pi\left[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right]$$

$$\sin(\omega_0 t) \leftrightarrow j\pi\left[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right]$$

خصائص تحويل فورييه Properties of the Fourier Transform

سوف نعرض في هذا الجزء النظريات الهامة في تحويل فورييه:

1) خاصية الخطية (نظرية التنضد):

Linearity Property (Superposition Theorem)

من أجل مجموع توابع زمنية يتحقق الآتي:

$$FT\{a_1.s_1(t)+a_2.s_2(t)\}=a_1.S_1(f)+a_2.S_2(f)$$

وبشكل عام: $$FT\left\{\sum_i a_i.s_i(t)\right\}=\sum_i a_i.S_i(f)$$

حيث a_i ثوابت وأن: $FT\{s_i(t)\}=S_i(f)$

أي أن تحويل فورييه لمجموع عدة توابع يساوي مجموع تحويلات فورييه لهذه التوابع.

2) خاصية التشابه أو خاصية التدريج الترددي Scaling in the Frequency Domain

إذا كان $FT\{s(t)\} = S(f)$ فأن: $FT\{s(bt)\} = \frac{1}{|b|}.S\left(\frac{f}{b}\right)$

حيثb ثابت عددي.

3) خاصية التدريج الزمني Scaling in the Time Domain : $FT\left\{\frac{1}{|b|}\cdot s\left(\frac{t}{b}\right)\right\} = S(bf)$

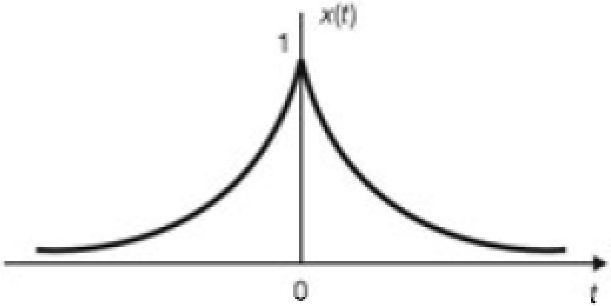
4) خاصية الإزاحة الزمنية Time Shifting Property:

$$FT\{s(t \pm t_o)\} = S(f).e^{\pm j2\pi f t_o}$$

5) خاصية الإزاحة الترددية Frequency Shifting Property: $FT\{s(t).e^{\pm j2\pi f_o t}\} = S(f \mp f_o)$

Find the Fourier transform of the signal [Fig. 5-18(*a*)]

$$x(t) = e^{-a|t|} \qquad a > 0$$



Signal $x(t)$ can be rewritten as

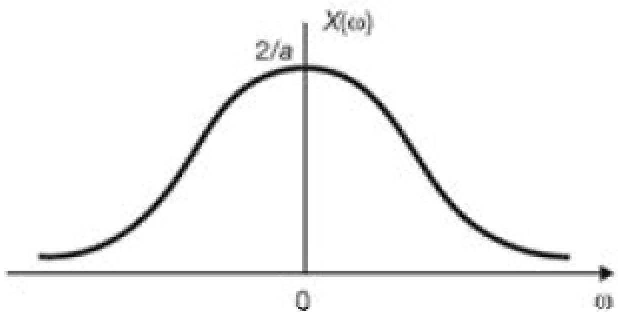
$$x(t) = e^{-a|t|} = \begin{cases} e^{-at} & t > 0 \\ e^{at} & t < 0 \end{cases}$$

$$X(\omega) = \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

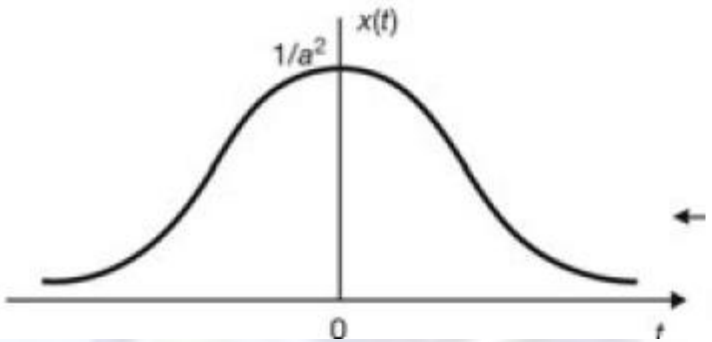
$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} = \frac{2a}{a^2 + \omega^2}$$

$$e^{-a|t|} \leftrightarrow \frac{2a}{a^2 + \omega^2}$$



Find the Fourier transform of the signal [Fig. 5-19(a)]

$$x(t) = \frac{1}{a^2 + t^2}$$



F. Duality (or Symmetry):

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

$$e^{-a|t|} \leftrightarrow \frac{2a}{a^2 + \omega^2}$$

Now by the duality property (5.54) we have

$$\frac{2a}{a^2 + t^2} \leftrightarrow 2\pi e^{-a|-\omega|} = 2\pi e^{-a|\omega|}$$

Dividing both sides by $2a$, we obtain

$$\frac{1}{a^2 + t^2} \leftrightarrow \frac{\pi}{a} e^{-a|\omega|}$$

Find the Fourier transforms of the following signals:

(a) $x(t) = 1$

(b) $x(t) = e^{j\omega_0 t}$

(c) $x(t) = e^{-j\omega_0 t}$

(d) $x(t) = \cos \omega_0 t$

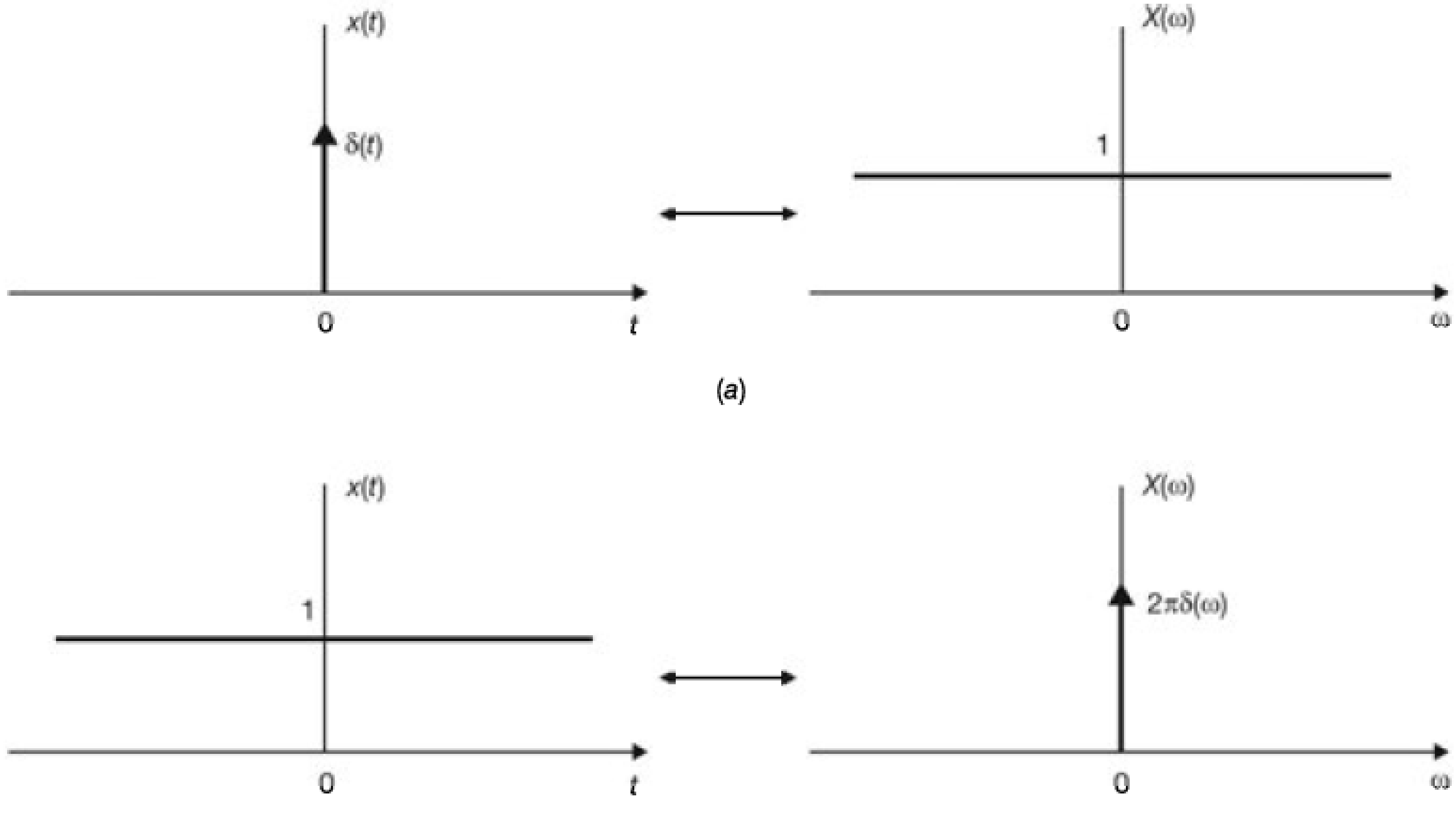
(e) $x(t) = \sin \omega_0 t$

(*a*) By Eq. (5.43) we have

$$\delta(t) \leftrightarrow 1 \qquad (5.1$$

Thus, by the duality property (5.54) we get

$$1 \leftrightarrow 2\pi\delta(-\omega) = 2\pi\delta(\omega) \qquad (5.1$$



(*b*)

(*b*) Applying the frequency-shifting property (5.51) to Eq. (5.141), we get

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

(c) From Eq. (5.142), it follows that

$$e^{-j\omega_0 t} \leftrightarrow 2\pi\delta(\omega + \omega_0) \quad (5.14$$

(d) From Euler's formula we have

$$\cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{j\omega_0 t})$$

Thus, using Eqs. (5.142) and (5.143) and the linearity property (5.49), we get

$$\cos \omega_0 t \leftrightarrow \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \quad (5.14$$

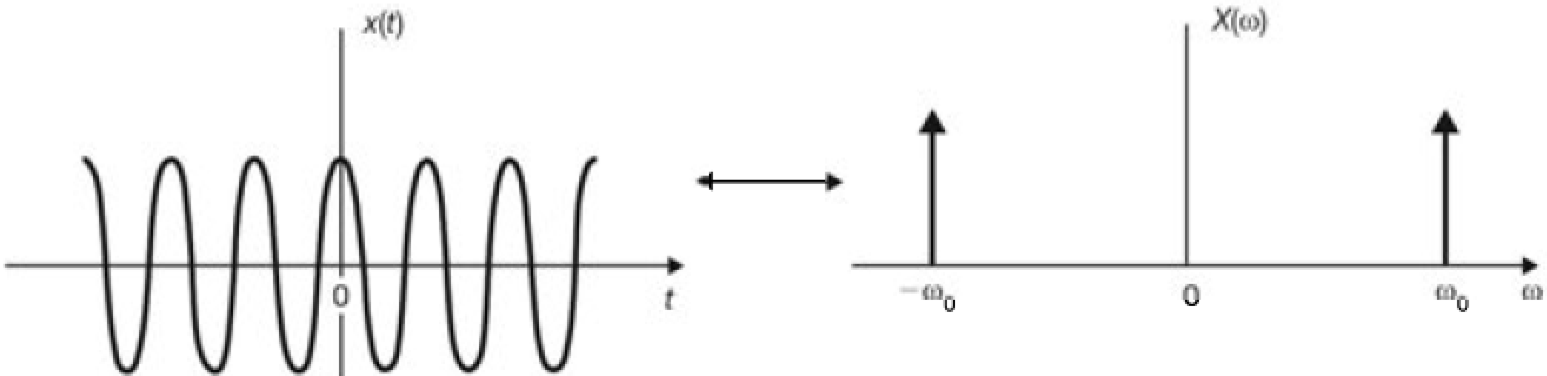
Fig. 5-21 illustrates the relationship in Eq. (5.144).

(e) Similarly, we have

$$\sin \omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$$

and again using Eqs. (5.142) and (5.143), we get

$$\sin \omega_0 t \leftrightarrow -j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] \quad (5.14$$



5.26. Show that

$$x(t)\cos\omega_0 t \leftrightarrow \frac{1}{2}X(\omega-\omega_0)+\frac{1}{2}X(\omega+\omega_0)$$

and

$$x(t)\sin\omega_0 t \leftrightarrow -j\left[\frac{1}{2}X(\omega-\omega_0)-\frac{1}{2}X(\omega+\omega_0)\right]$$

From Euler's formula we have

$$\cos\omega_0 t=\frac{1}{2}(e^{j\omega_0 t}+e^{-j\omega_0 t})$$

Then by the frequency-shifting property (5.51) and the linearity property (5.49), we obtain

$$\begin{aligned}\mathscr{F}[x(t)\cos\omega_0 \mathrm{t}] &= \mathscr{F}\left[\frac{1}{2}x(t)\,e^{j\omega_0 t}+\frac{1}{2}x(t)\,e^{-j\omega_0 t}\right]\\ &=\frac{1}{2}X(\omega-\omega_0)+\frac{1}{2}X(\omega+\omega_0)\end{aligned}$$

Hence,

$$x(t)\cos\omega_0 t \leftrightarrow \frac{1}{2}X(\omega-\omega_0)+\frac{1}{2}X(\omega+\omega_0)$$

In a similar manner we have

$$\sin \omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$$

and

$$\mathscr{F}[x(t)\sin \omega_0 t] = \mathscr{F}\left[\frac{1}{2j}x(t)\,e^{j\omega_0 t} - \frac{1}{2j}x(t)\,e^{-j\omega_0 t}\right]$$

$$= \frac{1}{2j}X(\omega - \omega_0) - \frac{1}{2j}X(\omega + \omega_0)$$

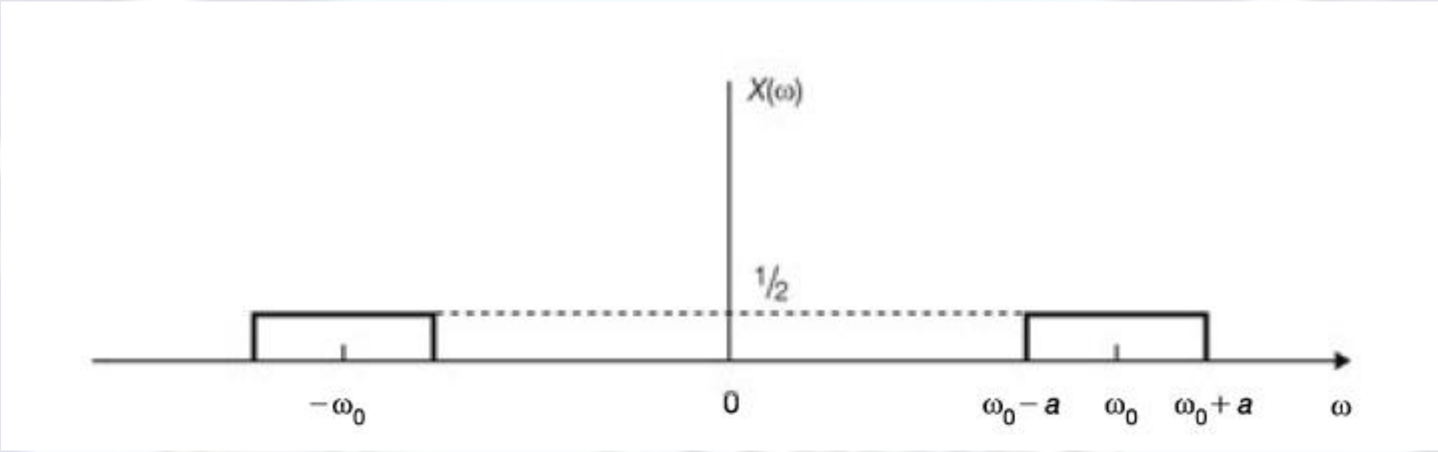
Hence,

$$x(t)\sin \omega_0 t \leftrightarrow -j\left[\frac{1}{2}X(\omega - \omega_0) - \frac{1}{2}X(\omega + \omega_0)\right]$$

5.27. The Fourier transform of a signal $x(t)$ is given by [Fig. 5-23(a)]

$$X(\omega) = \frac{1}{2} p_a(\omega - \omega_0) + \frac{1}{2} p_a(\omega + \omega_0)$$

Find and sketch $x(t)$.



$$x(t) = \frac{\sin at}{\pi t} \cos \omega_0 t$$